Localization and mobility edge for sparsely random potentials

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Abstract

In this paper we consider sparsely random potentials λV^ω, V^ω supported on a sparse subset S of $\mathbb{Z}^\nu$ and a bounded self-adjoint free part H_0 and show the presence of absolutely continuous spectrum and pure point spectrum for $H_0 + \lambda V^\omega$ when λ is large and $V^\omega(n)$ are independent random variables for n in S, with either identical distributions or distributions whose variance goes to ∞ with n, when $\nu \geq 5$.

1 Introduction

In this paper we consider random potentials on the ν dimensional lattice. The class of potentials considered in this paper are with sparse support but large disorder. A set S is sparse for us if the number of sites in any cube that belong to S grows at a fractional power of the volume of the cube, as the volume of the cube goes to ∞. The criterion of sparseness is stated in terms of the decay of the wave packets generated by the free evolution in our assumption on sparseness.

We are motivated by the difficult problem of existence of absolutely continuous spectrum in the higher dimensional Anderson model with small disorder and especially the question of whether a sharp mobility edge exists or not. We look for some models of random potentials that exhibit this behaviour.

In an earlier version of this paper we claimed that there is a sharp mobility edge for the large disorder models studied in this paper, but the argument we presented there had a gap.

However the techniques of Aizenman-Molchanov allow for the inclusion of a potential going to infinity at ∞, for which the claim on the mobility edge is true. This is our second theorem.

We list a part of the relevant literature on the Anderson model in the references for the benefit of the reader.

Once class of models considered in Krishna [18] were random potentials with decaying randomness with independent potentials at different sites, in higher dimensions, for which some absolutely continuous spectrum was shown to exist. For these class of models the question of mobility edge was considered in Kirsch-Krishna-Obermeit [16], and the answer was obtained for a class of potentials, when the decay rate is fast and the dimension is large.

In this paper we use the criterion from scattering theory for showing the existence of absolutely continuous spectrum and the technique of Aizenman-Molchanov [3] to verify the Simon-Wolff [24] criterion for the absence of continuous spectrum outside a band.

Firstly we assume that the unperturbed part satisfies the

Assumptions 1.1. *Let H_0 be a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^\nu)$, with*

$$\|H_0\|_s \equiv \sup_m \left(\sum_{n \in \mathbb{Z}^\nu} |\langle \delta_n, H_0 \delta_m \rangle|^s \right)^{1/s} < \infty \tag{1}$$

for all $s_0 < s < 1$ with $s_0 > 0$.

We note that in the case when $H_0 = \Delta$, the usual finite difference operator given by $(\Delta u)(n) = \sum_{|n-i|=1} u(i)$, we have $\|H_0\|_s = (2\nu)^{1/s}$. We also remark that once we assume the finiteness of the sum for s_0 it follows for all s, however we used the definition to fix the notation $\|H_0\|_s$.

Next we consider some subsets of $\mathbb{Z}^\nu$ which are regular in the following sense with respect to H_0. This criterion is useful in proving the existence of the wave operators later.

Definition 1.2. *Let S be any subset of $\mathbb{Z}^\nu$ and P_S the orthogonal projection on to the subspace $\ell^2(S)$ in $\ell^2(\mathbb{Z}^\nu)$. Then we call S* **sparse relative to** *H_0, a self-adjoint operator with non-empty absolutely continuous spectrum, whenever there exists a dense subset $\mathbb{D}$ contained in the absolutely continuous subspace of H_0 such that*

$$\int dt \|P_S exp\{-itH_0\}\phi\| < \infty, \quad \forall \phi \in \mathbb{D}. \tag{2}$$

If A is an operator of multiplication by a real sequence $\{a_n, n \in \mathbb{Z}^\nu\}$, then we say S is sparse relative to H_0 with weight A if,

$$\int dt \|AP_S exp\{-itH_0\}\phi\| < \infty, \quad \forall \phi \in \mathbb{D}. \tag{3}$$

Remark: 1. The assumption may not be satisfied even for finite sets S if H_0 is an arbitrary self-adjoint operator with non-empty absolutely continuous spectrum. For certain class of S with infinite cardinality and for $H_0 = \Delta$, and dimension $\nu \geq 4$, the H_0 sparseness was shown in Krishna [19]. Such sets S would be sparse in $\mathbb{Z}^\nu$. The class of subsets considered there are bigger than those considered as examples below.

2. One should contrast the sparseness criterion with the similar looking smoothness criterion widely used in scattering theory.

3. We note that there cannot be any non-zero operator H_0 on $\ell^2(\mathbb{Z}^\nu)$ with some absolutely continuous spectrum, such that $k\mathbb{Z}^\nu$ is sparse relative to it for any non-zero integer k.

Finally we assume that the random potential has sparse support and that its distribution has finite variance. The finiteness of the variance is needed in our proof of the presence of absolutely continuous spectrum for the perturbed operators.

Assumptions 1.3. *Let S be any subset of $\mathbb{Z}^\nu$. Let $V_S^\omega(n), n \in S$ be independent real valued random variables which are identically distributed according to an absolutely continuous probability distribution μ on $\mathbb{R}$ satisfying*

$$\sigma^2 \equiv \int |x|^2 d\mu(x) < \infty \quad \text{and} \quad \mu(a-\delta, a+\delta) \leq C\delta\mu(a-b, a+b), \quad \forall a \quad \text{and} \quad 0 \leq \delta < 1, \tag{4}$$

with C independent of a and δ for some $b \geq 1$.

(We denote by μ_0 the atomic probability measure on $\mathbb{R}$ giving mass 1 to the point 0. Then $V_S^\omega(n)$ are real valued measurable functions on a probability space $(\Omega, \mathbb{P})$, $\Omega = \mathbb{R}^S \times \mathbb{R}^{S^c}$ and $\mathbb{P} = \times_S(\mu) \times \times_{S^c}(\mu_0)$, so the parameter ω denotes a point in Ω – or equivalently a real valued sequence indexed by points of $\mathbb{Z}^\nu$ – and all our future references to a.e., are denoted with respect to this measure $\mathbb{P}$.)

Let V_S^ω denote the operator

$$(V_S^\omega u)(n) = V_S^\omega(n)\chi_S(n)u(n), \quad n \in \mathbb{Z}^\nu.$$

where χ_S denotes the indicator function of the set S. Let H_0 be some non random background bounded self-adjoint operator as in assumption (1.1). In the case when the measure μ has infinite support, the above are a family of unbounded self-adjoint operators having the set of finite vectors in their domain for almost every ω.

The main theorems of this paper are the following.

Theorem 1.4. *Let H_0 be any bounded self-adjoint operator satisfying assumption (1.1) and having some absolutely continuous spectrum. Let $S \subset \mathbb{Z}^\nu$ be sparse relative to H_0. Let $V_S^\omega(n)$ satisfy the assumption (1.3) and if μ has infinite support assume further that*

$$\sum_{m \in S} |(1+|m|)^\beta (e^{-itH_0}\phi)(m)|^2 < \infty, \phi \in \mathbb{D}, \tag{5}$$

for some $\beta > \nu$ and each fixed t. Consider the operator $H_\lambda^\omega = H_0 + \lambda V_S^\omega$. Then,

1. *$\sigma_{ac}(H_\lambda^\omega) \supset \sigma_{ac}(H_0)$ a.e. ω and*
2. *For each $0 < s < 1$, there is a $\lambda_s < \infty$, such that for any $\lambda > \lambda_s$,*

$$\sigma_c(H_\lambda^\omega) \subset [-\|H_0\|_s, \|H_0\|_s].$$

Remark: 1. The above theorem does not show the existence of spectrum outside $[-\|H_0\|_s, \|H_0\|_s]$. The existence of spectrum there can be shown for large λ on the lines of Kirsch-Krishna-Obermeit [16], even for the case when μ has compact support. The technique uses rank one perturbations to obtain a Weyl sequence.

2. The proof of the above theorem relies in part on the decoupling bounds obtained by Aizenman-Molchanov [3], where the constants $\lambda_s \to \infty$

as s approaches 1, so for any finite coupling constant λ, there is always an $s < 1$, with $\lambda < \lambda_s$. Therefore for any finite λ there is always a region $(\|H_0\|_1, \|H_0\|_s) \cup (-\|H_0\|_s, -\|H_0\|_1)$, where we cannot determine the spectral behaviour by this method. However when the potential goes to ∞ at ∞, while being supported on the sparse set, this problem can be avoided, which is our next theorem.

3. Recently considering surface randomness, Jaksic -Molchanov [20] proved pure point spectrum outside [-4,4] in the case when the randomness is on the boundary of $\mathbb{Z}^2_+$ and this is not a sparse set.

Theorem 1.5. *Let H_0 and V^ω_S be as in theorem (1.4). Let a_n be a sequence of positive numbers $|a_n| \to \infty$ as $|n| \to \infty$. Let A be the operator of multiplication by a_n and let $V^\omega_{S,A} = AV^\omega_S$. Assume further that S is sparse relative to H_0 with weight A. Consider $H^\omega = H_0 + AV^\omega_S$. Then,*

1. *$\sigma_{ac}(H^\omega) \supset \sigma_{ac}(H_0)$ a.e. ω and*
2. *The continuous spectrum satisfies*

$$\sigma_c(H^\omega) \subset [-\|H_0\|_1, \|H_0\|_1].$$

Remark: 1. In the case of operators H_0 the boundary of whose spectrum coincides with the points $\pm\|H_0\|_1$, they are the mobility edges . This happens for example for the operators Δ or some of the other examples given later.

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2 Proof of the theorems

Before we get to the proof of the theorems, we explain the motivation for considering the above model and the ideas involved briefly for the benefit of the reader. Our motivation comes from the effort to find random models that exhibit the expected behaviour of the spectrum of the Anderson model at small disorder. The reader who does not wish to be lost in the generality, could consider $H_0 = \Delta$ and work through the paper.

The general theory of scattering gives a way of checking if a given self-adjoint operator A has some absolutely continuous spectrum or not. The

technique is to find another self-adjoint operator B which is known to have some absolutely continuous spectrum and verify that the wave operators $S - \lim e^{-itA}e^{itB}P_{ac}(B)$ exist. Then it follows that $\sigma_{ac}(A) \supset \sigma_{ac}(B)$. We use this technique with B as the free operator H_0 of our model and A the random operator. To show the existence of the above limits is done via the Cook method of showing that the integral $\int \|(A-B)e^{itB}f\|dt$ is finite. The sparseness condition we imposed is an abstract condition that requires that the support (in $\mathbb{Z}^\nu$) of the potential is such as to make this integral finite, for a set of f dense in the absolutely continuous spectral subspace of B.

Our class of examples come from operators of multiplication by real valued functions on $[0, 2\pi]^\nu$, so that they commute with Δ and also satisfy some smoothness condition so as to make the matrix elements $H_0(n,m)$ in $\ell^2(\mathbb{Z}^\nu)$ decay at a required rate, in addition to giving decay in t for the function $e^{itB}(n,m)$, obtained via a stationary phase argument.

We impose the condition on the support of the random potential based on the behaviour of the integrals $(e^{itB}f)(m)$ as a function of m and t, so as to make them sparse relative to B *á la* the sparseness assumption.

As for the pure point spectrum outside the smallest interval containing the absolutely continuous spectrum of our models, we use the Aizenman-Molchanov technique. The idea behind the method is to use the presence of randomness with large coupling to obtain decay estimates on the averages of small moments of the resolvent kernels of the random operators. Aizenman-Molchanov estimates use the fact that the potentials have components which are mutually independent at different sites and also the regularity of their distribution. However this technique fails at the points in $\mathbb{Z}^\nu$, where the random potential is absent. So we modify the method, and use their estimates at sites in the lattice where the potential is non-zero and at the sites where the potential is zero, we use the fact that $|E|$ is large. Thus we use both the large λ and large $|E|$ conditions in obtaining the exponential decay estimates on the Green functions.

Proof of theorem (1.4): As for item (1) of the theorem we prove that the wave operators namely

$$\text{S-lim} \;\; exp\{iH_\lambda^\omega t\}exp\{-iH_0 t\}P_{ac}(H_0) \tag{6}$$

exist, where $P_{ac}(H_0)$ denotes the orthogonal projection onto the absolutely continuous spectral subspace of H_0. That this implies (1) is standard, see [18]).

Suppose we have, $\int \|V_S^\omega e^{-itH_0}\phi\|dt < \infty$ for almost every ω for all $\phi \in \mathbb{D}$, then the sequence $e^{itH_\lambda^\omega}e^{-itH_0}\phi$ is Cauchy for all $\phi \in \mathbb{D}$, from which the existence of the wave operators follows, since $\mathbb{D}$ is dense in the absolutely continuous spectral subspace of H_0, by assumption.

We take β as in the theorem and consider the sets

$$A_m = \{\omega : |V^\omega(m)| > (1+|m|)^\beta\}, \quad m \in S.$$

Then by assumption (1.3), on the variance of μ, we have

$$\sum_{m\in S} \mathbb{P}(A_m) \leq \sum_{m\in S} \int_{A_m} \frac{1}{x} x d\mu(x) \leq \sigma \sum_{m\in\mathbb{Z}^\nu} 1/(1+|m|)^\beta < \infty,$$

using the Cauchy-Schwarz inequality. Therefore by Borel-Cantelli lemma, $\Omega_0 = \{\omega : \sum_{m\in S} |m|^{2\beta}|u(m)|^2 < \infty, \ \text{ for all } \ u \in Dom(V_S^\omega)\}$ has probability 1.

Then it follows from the assumption in the theorem, equation (5), that the random variable $\|V_S^\omega e^{-itH_0}\phi\|$, is defined finitely on Ω_0 for each t fixed.

The square of this random variable is integrable in ω for each fixed t, as can be seen from the following estimate, H_0 sparseness of S and use of Fubini's theorem.

$$\begin{aligned} & \mathbb{E}\{\|V_S^\omega e^{-itH_0}\phi)\|^2\} \\ & \sum_{m\in S} \mathbb{E}\{|V^\omega(m)|^2 |e^{-itH_0}\phi)(m)|^2\} \\ \leq \sum_{m\in S} \mathbb{E}\{|V^\omega(m)|^2\} & |e^{-itH_0}\phi)(m)|^2 \leq \sigma^2 \sum_{m\in S} |e^{-itH_0}\phi)(m)|^2. \end{aligned} \tag{7}$$

Therefore its average value is also integrable in t as can be seen from the inequalities,

$$\begin{aligned} \int dt \mathbb{E}\{\|V_S^\omega e^{-itH_0}\phi\|\} & \leq \int dt \{\mathbb{E}\|V_S^\omega e^{-itH_0}\phi\|^2\}^{1/2} \\ & \leq \sigma \int dt \|P_S e^{-itH_0}\phi\| < \infty \end{aligned} \tag{8}$$

with the last inequality resulting by assumption of the H_0 sparseness of S. Then using Fubini, we conclude that

$$\mathbb{E}\{\int dt \|V_S^\omega e^{-itH_0}\phi\|\} < \infty$$

showing that $\|V_S^\omega e^{-itH_0}\phi\|$ is integrable in t for almost all ω, proving the result.

The proof of (2) of the theorem follows from the lemma (2.1) below.

In the following we denote by $G(E+i\epsilon, n, m) = \langle \delta_n, (H_\lambda^\omega - E - i\epsilon)^{-1}\delta_m\rangle$.

Lemma 2.1. *Suppose H_λ^ω be an operator as in theorem (1.4). Then for any $s_0 < s < 1$, there is a $\lambda_s > 0$ such that for any $\lambda > \lambda_s$ and a.e. ω we have $\sigma_c(H_\lambda^\omega) \subset [-\|H_0\|_s, \|H_0\|_s]$.*

Proof: We prove this lemma by proving that there is a λ_s such that for each $s_0 < s < 1$, the estimate

$$\sum_{m\in\mathbb{Z}^\nu} \mathbb{E}\{|G(E+i\epsilon,n,m)|^s\} \le C < \infty \tag{9}$$

is valid whenever $|E| > \|H_0\|_s$ and $\lambda > \lambda_s$ with C independent of ϵ. This estimate implies by integrating over E in an interval $[a,b] \subset (-\infty, -\|H_0\|_s) \cup (\|H_0\|_s, \infty)$ and using the fact that $(\sum x_i)^s \le \sum x_i^s$ for $x_i \ge 0$ and $0 < s < 1$,

$$\begin{aligned} &\int_a^b dE \ \ \mathbb{E}\{\sum_{m\in\mathbb{Z}^\nu} |G(E+i\epsilon,n,m)|^2\}^{s/2} \le \\ &\int_a^b dE \ \ \mathbb{E}\{\sum_{m\in\mathbb{Z}^\nu} |G(E+i\epsilon,n,m)|^s\} < \infty. \end{aligned} \tag{10}$$

Hence for a.e. $(\omega, E) \in \Omega \times [a,b]$, we have

$$\sum_{m\in\mathbb{Z}^\nu} |G(E+i0,n,m)|^s < \infty \ \text{ and hence } \ \sum_{m\in\mathbb{Z}^\nu} |G(E+i0,n,m)|^2 < \infty.$$

by means of Fatou's lemma and the existence of the limit $\lim \epsilon \downarrow 0 \sum_{m\in\mathbb{Z}^\nu} |G(E+i\epsilon,n,m)|^2$. Therefore by the Simon-Wolff [24] criterion, the spectral measure of the operator H_λ^ω associated with the vector δ_n has no continuous component supported in $[a,b]$. Since this happens for all $n \in \mathbb{Z}^\nu$ and since the collection $\{\delta_n\}$ forms an orthonormal basis in $\ell^2(\mathbb{Z}^\nu)$ as n varies in $\mathbb{Z}^\nu$, it follows that $\sigma_c(H_\lambda^\omega) \cap [a,b] = \emptyset$, for almost all points in Ω. By taking a countably many bounded intervals we see that this implies that for almost all points in Ω, $\sigma_c(H_\lambda^\omega) \subset [\|H_0\|_s, \|H_0\|_s]$.

Therefore we prove the estimate in equation (9), to do which we fix some s in $(s_0, 1)$ and consider the equation

$$\begin{gathered}(\lambda V_S^\omega(m) - E - i\epsilon)G(E+i\epsilon, n, m) + \\ \sum_{k\in\mathbb{Z}^\nu} \langle \delta_k, H_0\delta_m\rangle G(E+i\epsilon, n, k) = \delta_{n.m}.\end{gathered} \tag{11}$$

We transfer the sum involving H_0 to the right hand side and take the average of the absolute value raised to power s to get the inequality, (using the fact that $(\sum x_i)^s \leq \sum x_i^s$ for $x_i \geq 0$ and $0 < s < 1$),

$$\begin{gathered}\mathbb{E}\{|(\lambda V_S^\omega(m) - E - i\epsilon)G(E+i\epsilon, n, m)|^s\} \leq \\ \delta_{n,m} + \sum_{k\in\mathbb{Z}^\nu} |\langle \delta_k, H_0\delta_m\rangle|^s \mathbb{E}\{|G(E+i\epsilon, n, k)|^s\}.\end{gathered} \tag{12}$$

In the inequality below we set

$$C(E, \lambda, s) = \begin{cases} |E|^s, & m \notin S \text{ and} \\ C(\lambda, s) & m \in S \end{cases}$$

where $C(\lambda, s) = |\lambda|^s (1-s)^s D(s)$ is the constant appearing in proposition (4.1). Therefore when $|E| > \|H_0\|_s$ and $\lambda > \lambda_s (= \|H_0\|_s/(1-s)D(s)^{1/s})$, we can make $C(E, \lambda, s) > \|H_0\|_s^s$.

Now we use the decoupling principle (proposition (4.1)) and Fubini to interchange the sum and the integral to get,

$$\begin{gathered}C(E, \lambda, s)\mathbb{E}\{G(E+i\epsilon, n, m)|^s\} \leq \\ \mathbb{E}\{|(\lambda V_S^\omega(m) - E - i\epsilon)G(E+i\epsilon, n, m)|^s\} \leq \\ \delta_{n,m} + \sum_{k\in\mathbb{Z}^\nu} |\langle \delta_k, H_0\delta_m\rangle|^s \mathbb{E}\{|G(E+i\epsilon, n, k)|^s\}.\end{gathered} \tag{13}$$

Then by our choice of the λ, $C(E, \lambda, s) > \|H_0\|_s^s$, so using the proposition below on the bounds on $\mathbb{E}\{|G(E+i\epsilon, n, m)|^s\}$, uniform in ϵ, the proof now follows on the same lines of Aizenman-Molchanov [3], by repeating the above estimate $|n-m|$ times to get the following bound, where we set

$$\begin{gathered}K_s^{*j}(n) = \sum_{i_1, \cdots, i_{j-1}, m \in \mathbb{Z}^\nu} \frac{|\langle \delta_n,\ H_0\delta_{i_1}\rangle|^s |\langle \delta_{i_1},\ H_0\delta_{i_2}\rangle|^s \cdots |\langle \delta_{i_{j-1}},\ H_0\delta_m\rangle|^s}{C(E, \lambda, s)^j} \\ \text{and} \quad k_s = \|H_0\|_s^s / C(E, \lambda, s),\end{gathered} \tag{14}$$

then $\sup_n K_s^{*j}(n) \leq k_s^j$.

$$\begin{aligned}\sum_{m \in \mathbb{Z}^\nu} \mathbb{E}\{G(E+i\epsilon,n,m)|^s\} &\leq 1 + \sum_{j=1}^{\infty} K_s^{*j}(n) + \sum_{m \in \mathbb{Z}^\nu} K_s^{|n-m|} D(E,\lambda,s) \\ &\leq \sum_{j=0}^{\infty} k_s^j + (\sum_{l=1}^{\infty} l^{\nu-1} k_s^l) D(E,\lambda,s) < \infty,\end{aligned} \tag{15}$$

since $k_s < 1$ by assumption on E and λ, with the bound independent of ϵ. (We note that one could have used a Combes-Thomas type argument to avoid using the uniform bounds provided, the quantities $|\langle \delta_n H_0 \delta_m \rangle|$ have exponential decay in $|n-m|$ as it happens for Δ and other examples with finite range off diagonal parts.)

The uniform bounds below are analogous to the uniform bounds obtained by Aizenman-Molchanov [3](equation (2.12)). (The following proposition uses ideas similar to the Wegner estimate of Kirsch [15] or Obermeit [21] in the proof of localization.)

Proposition 2.2. *Consider the operator H_λ^ω as in theorem (1.4) or H^ω as in (1.5) and let s_0 as in assumption (1.1).*

1. *In the case of theorem (1.4), for all $E \in \mathbb{R} \setminus \sigma(H_0)$ and $s_0 < s < 1$,*
$$\mathbb{E}\{|G(E+i\epsilon,n,m)|^s\} \leq D(E,\lambda,s) < \infty. \tag{16}$$

2. *In the case of theorem (1.5), for all $E \in \mathbb{R} \setminus \sigma(H_0)$ and $s_0 < s < 1$,*
$$\mathbb{E}\{|G(E+i\epsilon,n,m)|^s\} \leq D(E,\sqrt{|a_m a_n|},s) < \infty. \tag{17}$$

The constants $D(E,\cdot,s)$ appearing above are uniformly bounded in E, for E in any compact subset of $\mathbb{R} \setminus [-\|H_0\|_s, \|H_0\|_s]$.

Proof: We split the proof of part (1) of the proposition in to three cases. The proof of part (2) of the theorem is similar, by replacing – for each $n, m \in S$ – λ by $\sqrt{|a_m a_n|}$ (which is the form in which the estimate of equation (18) is valid, see for example [16], where the estimate was shown to be valid with λ replaced by a_n and a_m seperately, so by interpolation the present estimate comes out), in the estimate of Case 1, below and going through the proof of all the cases.

Case 1: n, m $\in S$. Then the bound

$$\mathbb{E}\{|G(E+i\epsilon,n,m)|^s\} \le (2\sqrt{2})^s/(|\lambda|^s(1-s)) \tag{18}$$

is similar to the estimate proved using Theorem II.1 Aizenman-Molchanov [3]. We designate the constant on the right hand side of the above inequality as $D_0(E,\lambda,s)$.

Case 2: $n \in S$ and $m \in S^c$ or $n \in S^c$ and $m \in S$. We consider the possibility $n \in S^c$ and $m \in S$, the proof of the other possibility is similar. Let $H_{0,S^c} = P_{S^c} H_0 P_{S^c}$, then $\|H_{0,S^c}\| \le \|H_0\|_s$. We then consider the operator $H_\lambda^\omega(S) = H_{0,S^c} + V_S^\omega$, set $z = E + i\epsilon$ and use the resolvent equation to write

$$\begin{aligned}(H_\lambda^\omega - z)^{-1}(n,m) &= (H_\lambda^\omega(S) - z)^{-1}(n,m) \\ &- \sum_{k\in S^c, l\in S} (H_\lambda^\omega(S) - z)^{-1}(n,k) \\ [(H_0 - H_{0,S^c})(k,l)]&(H_\lambda^\omega(S) - z)^{-1}(l,m)\end{aligned} \tag{19}$$

where we have used the fact that since $n \in S^c$, when $k \in S$, $(H_{0,S^C} - z)^{-1}(n,k) = 0$ and hence $(H_\lambda^\omega(S) - z)^{-1}(n,k) = 0$. Then it follows that for any $s_0 < s < 1$, the estimate

$$\begin{aligned}|(H_\lambda^\omega - z)^{-1}(n,m)|^s &\le |(H_\lambda^\omega(S) - z)^{-1}(n,m)|^s \\ &+ \sum_{k\in S^c, l\in S} |(H_\lambda^\omega(S) - z)^{-1}(n,k)|^s \\ |[(H_0 - H_{0,S^c})(k,l)]|^s&|(H_\lambda^\omega(S) - z)^{-1}(l,m)|^s\end{aligned} \tag{20}$$

is valid. Observe that since $k \in S^c$ and $l \in S$, we have

$$(H_\lambda^\omega(S) - z)^{-1}(k,l) = (H_{0,S^c} - z)^{-1}(k,l),$$

since by assumption (1.1) $|E| > \|H_0\|_s$ hence $|E| > \|H_{0,S^c}\|_s$. Therefore the estimates of proposition (4.3) are valid and after taking averages in the above inequality we have

$$\begin{aligned}\mathbb{E}\left\{|(H_\lambda^\omega - E - i\epsilon)^{-1}(n,m)|^s\right\} &\le |(H_{0,S^c} - E - i\epsilon)^{-1}(n,m)|^s \\ &+ \sum_{k\in S^c, l\in S} |(H_{0,S^c} - E - i\epsilon)^{-1}(n,k)|^s \\ |[(H_0 - H_{0,S^c})(k,l)]|^s &\mathbb{E}\left\{|(H_\lambda^\omega(S) - E - i\epsilon)^{-1}(l,m)|^s\right\} \\ &\le \frac{1}{dist([-\|H_0\|_s, \|H_0\|_s, E)} \\ +\|H_0\|_s^s C(H_{0,S^c}, E, s) sup_{l\in S} &\mathbb{E}\left\{|(H_\lambda^\omega(S) - E - i\epsilon)^{-1}(l,m)|^s\right\}\end{aligned} \tag{21}$$

where the constant $C(H_{0,S^C}, E, s)$ is given by proposition (4.3), and is finite for each $E \in \mathbb{R} \setminus [-\|H_0\|_s, \|H_0\|_s]$, fixed. Since now both l, m are in S, we can use the estimate in Case 1 to conclude that,

$$\begin{aligned} &\mathbb{E}\left\{|(H^\omega_\lambda - E - i\epsilon)^{-1}(n,m)|^s\right\} \\ \leq\ & 1/dist([-\|H_0\|_s, \|H_0\|_s], E) + \|H_0\|_s^s C(H_{0,S^c}, E, s) D_0(E,\lambda,s) \end{aligned} \tag{22}$$

We designate the quantity on the right hand side of the above inequality as $D_1(E,\lambda,s)$.

Case 3: In case 2, we started with a uniform bound valid for the average of the s-th moment when the sites were both on S, to get a E dependent bound, but uniform in m, n when at least one of them is in S. We thus could relax the condition on m,n at the cost of having the bounding constant depend on E. It is clear that we can repeat this trick, to cover all sites m, n in $\mathbb{Z}^\nu$. Therefore using the result proved in Case 2 above we repeat the proof of case 2 when $n, m \in S^c$ and we set the resulting constant in the inequality as $D_2(E,\lambda,s)$. Now we take

$$D(E,\lambda,s) = max\left\{D_0(E,\lambda,s), D_1(E,\lambda,s), D_2(E,\lambda,s)\right\}.$$

With this constant the proposition is valid. Further we see from the proof that each of the $D_i(E,\lambda,s)$ is uniformly bounded in any compact subset of $\rho(H_0)$ hence $D(E,\lambda,s)$ also satisfies this property.

Remark: The estimate in the above proposition did not depend upon the set S, so if we set the potential to be zero at some, or even all, of the sites in S, the result is still valid with the same bound. Of course, this amounts to shrinking the set S and when $S = \emptyset$, then the trivial uniform bound in terms of the inverse of the distance of E to $[-\|H_0\|_s, \|H_0\|_s]$ is valid.

Proof of theorem (1.5): The proof of item (1) of this theorem proceeds similar to that of item (1) in the earlier theorem. For that proof to go through we need that

$$\sigma \quad \int dt \|AP_S e^{-itH_0}\phi\| < \infty$$

which we ensured by assumption of sparseness of S relative to H_0 with weight A.

(2) We prove this part on the absence of continuous spectrum outside $[-\|H_0\|_1, \|H_0\|_1]$, by using the estimates of Aizenman-Molchanov. We do the proof in two steps. Step one consists of noting that if the average Green function is bounded for each E, then the sum of any finite number of them is

also bounded. Therefore we need to look at the decay of the average Green function outside a finite set of sites. We determine the finite set based on the number $0 < s < 1$ and proceed to show exponential decay on the complement of that set.

Let $\Lambda_s(n)$ be the smallest cube centered at n such that

$$B = \inf_{m \in \Lambda_s(n)^c \cap S} |a_m|^s (1-s)^s D(s) / \|H_0\|_s^s > 1.$$

where $D(s)$ is the constant appearing in the proposition (4.2). Since $|a_m| \to \infty$ as $|m| \to \infty$, such a cube exists for each fixed s in (0, 1) and each fixed n $\in \mathbb{Z}^\nu$. Then for each $m \in \Lambda_s(n)^s \cap S$ we have the estimate, as in the proof of theorem (1.4)(2),

$$\begin{aligned} B\|H_0\|_s^s \mathbb{E}\{|G(E+i\epsilon, n, m)|^s\} \leq \\ |a_m|^s (1-s)^s D(s) \mathbb{E}\{|G(E+i\epsilon, n, m)|^s\} \leq \\ \mathbb{E}\{|(a_m V^\omega(m)\chi_S(m) - E - i\epsilon) G(E+i\epsilon, n, m)|^s\} \leq \\ \delta_{n,m} + \sum_{k \in \mathbb{Z}^\nu} |\langle \delta_k, H_0 \delta_m \rangle|^s \mathbb{E}\{|G(E+i\epsilon, n, k)|^s\}. \end{aligned} \tag{23}$$

and for E in any compact subset of $\mathbb{R} \setminus [-\|H_0\|_s, \|H_0\|_s]$,

$$\begin{aligned} |E|^s \mathbb{E}\{|G(E+i\epsilon, n, m)|^s\} \leq \\ \mathbb{E}\{|(-E - i\epsilon) G(E+i\epsilon, n, m)|^s\} \leq \\ \delta_{n,m} + \sum_{k \in \mathbb{Z}^\nu} |\langle \delta_k, H_0 \delta_m \rangle|^s \mathbb{E}\{|G(E+i\epsilon, n, k)|^s\}. \end{aligned} \tag{24}$$

for all $m \in \Lambda_s(n)^c \setminus S$. Combining these two estimates we have that for any $m \in \Lambda_s(n)^c$,

$$\begin{aligned} C(E,s) \mathbb{E}\{|G(E+i\epsilon, n, m)|^s\} \leq \\ \mathbb{E}\{|(a_m V_S^\omega(m) - E - i\epsilon) G(E+i\epsilon, n, m)|^s\} \leq \\ \delta_{n,m} + \sum_{k \in \mathbb{Z}^\nu} |\langle \delta_k, H_0 \delta_m \rangle|^s \mathbb{E}\{|G(E+i\epsilon, n, k)|^s\}. \end{aligned} \tag{25}$$

where we have made use of the fact that outside S, $V_S^\omega = 0$ and have set

$$C(E,s) = \begin{cases} |E|^s, & m \notin S \text{ and} \\ B\|H_0\|_s^s & m \in S \cap \Lambda_s^c. \end{cases}$$

Then by assumption on E and definition of B we have that $C(E,s) > \|H_0\|_s^s$. Using this fact and the proposition (4.4) we have the inequality, for each $\epsilon > 0$,

$$
\begin{aligned}
C(E,s) \sum_{m \in \Lambda_s(n)^c} \mathbb{E}\{|G(E+i\epsilon,n,m)|^s\} \leq \\
\sum_{m \in \Lambda_s(n)^c} \delta_{n,m} + \sum_{m \in \Lambda_s(n)^c} \sum_{k \in \Lambda_s(n)^c} |\langle \delta_k, H_0 \delta_m \rangle|^s \mathbb{E}\{|G(E+i\epsilon,n,k)|^s\} \\
+ \sum_{m \in \Lambda_s(n)^c} \sum_{k \in \Lambda_s(n)} |\langle \delta_k, H_0 \delta_m \rangle|^s \mathbb{E}\{|G(E+i\epsilon,n,k)|^s\}.
\end{aligned}
\tag{26}
$$

Using this estimate, we get the bound,

$$
\begin{aligned}
\sum_{m \in \Lambda_s^n} \mathbb{E}\{G(E+i\epsilon,n,m)|^s\} \\
\leq \sum_{j=0}^{\infty} k_s^j + (\sum_{l=1}^{\infty} l^{\nu-1} k_s^l)(\sup_m D(E, \sqrt{a_n a_m}, s))(1+L_s(n)) < \infty,
\end{aligned}
\tag{27}
$$

where K_s, k_s are defined as in the equation (14) with the constant $C(E,s)$ given above replacing $C(E,\lambda,s)$ there and the number $L_s(n)$ is the cardinality of the set $\Lambda_s(n)$, which is finite by the assumption on $\{a_n\}$ and so is the number $\sup_m D(E, \sqrt{a_n a_m}, s)$. The above sum converges by the assumption that $C(E,s) > \|H_0\|_s^s$, so that $k_s < 1$.

From this estimate we conclude, as in the earlier theorem, that

$$\mathbb{P}\{\omega : \sigma_c(H^\omega) \subset [-\|H_0\|_s, \|H_0\|_s\} = 1.$$

So taking a sequence $s_k \uparrow 1$, we see that with probability 1,

$$\sigma_c(H^\omega) \subset \cap_k [-\|H_0\|_{s_k}, \|H_0\|_{s_k}] = [-\|H_0\|_1, \|H_0\|_1]$$

since a countable intersection of sets of measure 1 also has measure 1.

3 Examples

In this section we present a general class of examples of operators H_0 and subsets S of $\mathbb{Z}^\nu$ that satisfy our assumptions. The examples for H_0 comes from the spectral representation of Δ.

Let H_0 be the operator of multiplication by a function h in the spectral representation of Δ, where

Assumptions 3.1. *1. h is a real valued $C^{2\nu+2}$ function on $[0,2\pi]^\nu$ with*

$$C_h \equiv \sup_\alpha \sup_\theta |\frac{\partial^\alpha}{\partial\theta^\alpha} h(\theta)| < \infty \tag{28}$$

where α is a multi index $(\alpha_1, \cdots, \alpha_\nu)$ with $\alpha_i \geq 0$, $\sum \alpha_i = 2\nu + 2$. Assume further that

$$h^{(\alpha)}(\theta_1, \cdots, \theta_{i-1}, 0, \theta_{i+1}, \cdots, \theta_\nu) = h^{(\alpha)}(\theta_1, \cdots, \theta_{i-1}, 2\pi, \theta_{i+1}, \cdots, \theta_\nu)$$

for each i=1,..,ν and each multi index α with $|\alpha_i| \leq 2\nu + 2$.

2. h is separable, i.e. $h(\theta_1, \cdots, \theta_\nu) = \sum_{i=1}^\nu h_i(\theta_i)$.

3. For each $i = 1, \cdots, \nu$, $\frac{d^3}{d\theta_i^3} h_i(\theta_i) \neq 0$ whenever θ_i is a zero of $\frac{d^2}{d\theta_i^2} h_i(\theta_i) = 0$ whose number is assumed to be finite.

Remark: It may appear strange to the reader that we need the separability condition (2) above. The reason is that the higher dimensional version of the proposition (4.4) (on stationary phase) is not known when the second derivative of the phase function is singular at some point.

Proposition 3.2. *Let h be a function as in assumption (3.1)(1). Fix any $s_0 > \nu/2\nu + 1$. Then there is a constant $C(s_0, C_h)$ such that for any $s_0 < s$,*

$$\sup_{n\in\mathbb{Z}^\nu} \sum_{m\in\mathbb{Z}^\nu} |\langle \delta_n, H_0\delta_m\rangle|^s < C(s_0, C_h).$$

Proof: Writing the expression for $\langle \delta_n, H_0\delta_m\rangle$ in the spectral representation for H_0 we have

$$\langle \delta_n, H_0\delta_m\rangle = \frac{1}{(2\pi)^\nu} \int_{[0,2\pi]^\nu} e^{-i\sum_{j=1}^\nu (n-m)_j\theta_j} h(\theta_1, \cdots, \theta_\nu) \prod_{j=1}^\nu d\theta_j \tag{29}$$

Now using assumption (3.1)(1), integration by parts $(2\nu + 1)$ times with respect to the co-ordinate θ_i which is chosen such that $(n-m)_i \geq |n-m|/\nu$, gives the crude estimate

$$|\langle \delta_n, H_0\delta_m\rangle| \leq C_h \frac{\nu^{2\nu+1}}{|n-m|^{2\nu+1}}, \quad n \neq m. \tag{30}$$

This implies the proposition. Here the assumption on the derivatives at the boundary are made so that the boundary terms in the integration by parts vanish at each stage. In the following proposition we denote by $\|h'\| = \sup_i \sup_{\theta\in[0,2\pi]} |h_i'(\theta)|$.

Proposition 3.3. *Let h be a function satisfying assumption (3.1). Then we have the following estimates.*

1. $|\langle \delta_n, e^{\{-itH_0\}}\delta_m\rangle| \leq C/|n-m|^{2\nu+1}$, *if* $\nu|t|\|h'\|/|n-m| \leq 1/2$,
2. $|\langle \delta_n, e^{\{-itH_0\}}\delta_m\rangle| \leq C/|t|^{\nu/3}, |t| \geq t_0$, *for some* t_0 *large where* C, t_0 *are independent of n,m.*

Proof: The proof of the first estimate is a repeated integration by parts (see Stein, [25], VIII.1.3. Proposition 1) applied to one of the integrals in the product

$$\langle \delta_n, e^{\{-itH_0\}}\delta_m\rangle = \prod_{i=1}^{\nu} \frac{1}{2\pi} \int d\theta e^{-ith_i(\theta)+i(n-m)_i\theta}. \tag{31}$$

We use that integral (in the product) for which $|(n-m)_i| \geq |n-m|/\nu$ to do the integration by parts. Our assumption on the equality of the derivatives at the boundaries ensures that the boundary terms vanish for up to $2\nu+1$ derivatives, while the condition on t and $n-m$ ensures that $|1-th_i'(\theta)/(n-m)_i| > 1/2$, for all $\theta \in [0, 2\pi]$. From this lower bound the estimate $|[th_i'(\theta)] - (n-m)_i| \geq |n-m|/2\nu$ is clear and that this implies the estimate is straight forward.

To get the second estimate, we consider one of the integrals , say the one corresponding to the index i, in the product in equation (31) and obtain a $C/t^{(1/3)}$ bound for all $t > t_0$, the estimates for the other integrals is similar, resulting in the stated bound of the Proposition.

We know by assumption (3.1)(3), that the number of points in [0, 2π] where the second derivative of h_i vanishes is finite, say $x_1, \cdots, x_N$ and at these points the third derivative does not vanish. We also know from assumption (3.1), that h_i is $C^{2\nu+2}$. So we take any x in [0, 2π] and expand h_i about x using the Taylors formula with reminder to get

$$h_i(y) = \sum_{j=0}^{M} \frac{h_i^{(j)}(x)}{j!}(x-y)^j + R_i^{(M)}(y), \quad 0 \leq M \leq \nu+1,$$

for y in a neghbourhood of x. We then consider the sets

$$S_1(x) = \left\{ y \in [0, 2\pi] : |R_i^{(2)}(y)| < |h_i^{(2)}(x)|/2 \right\}.$$

when $x \notin \{x_1, \cdots, x_N\}$ and

$$S_1(x_j) = \left\{ y \in [0, 2\pi] : |R_i^{(3)}(y)| < |h_i^{(3)}(x_j)|/2 \right\}, j = 1, \cdots, N.$$

Each of the sets $S_1(x)$ is relatively open in $[0, 2\pi]$, by the continuity of the reminder terms $R_i^{(k)}$, k = 2,3. So one can find neighbourhoods $S(x)$ of x so that $\overline{S(x)} \subset S_1(x)$. Clearly $\cup_{x \in [0,2\pi]} S(x)$ covers the (compact) set $[0, 2\pi]$. Therefore, a finite collection of the above sets $S(x)$ cover $[0, 2\pi]$. Let $S(\alpha_j), j = 1, \cdots, M$ cover $[0, 2\pi]$. It is possible that some of the points α_j will correspond to some x_k at which the second derivative of h_i vanishes. Let us index the α_j such that $\alpha_j \in \{x_k, k = 1, \cdots, N\}, j = 1, \cdots, K$ ($K \leq N$) and the remaining α_j's are points where the second derivative of h_i does not vanish. Let ψ_j be a partition of unity subordinate to the cover $S(\alpha_j), j = 1, \cdots, M$.

Then,

$$\int_0^{2\pi} d\theta e^{-ith_i(\theta) + i(n-m)_i\theta} = \sum_{j=1}^{M} \int_0^{2\pi} d\theta e^{-ith_i(\theta) + i(n-m)_i\theta} \psi_j(\theta) \tag{32}$$

Suppose for the index j, the support of ψ_j is contained in $(0, 2\pi)$. Then the estimate for the integral $\int_0^{2\pi} d\theta e^{-ith_i(\theta)+i(n-m)_i\theta} \psi_j(\theta)$ follows from the proposition (4.4) where we set $\lambda = t$ and $\phi(\theta) = h_i(\theta) + ((n-m)_i/t)\theta$. We note that since the second and third derivatives of the ϕ above are independent of t, the proposition is still applicable, even though it seems that á priori ϕ has a "λ" dependence. Then we get $C/|t|^{1/3}$ bound for $j = 1, \cdots, K$ and $C/|t|^{1/2}$ bound for the remaining js, for large enough $|t|$.

It is in general possible for an arbitrary h_i, integration by parts leaves non-zero boundary terms at the points 0 and 2π, so we deal with this case separately.

Suppose, ψ_{j_1} and ψ_{j_2} are the functions which have the points 0 and 2π, respectively, in their support. Then, we first observe that we could have chosen the sets $S(\alpha_{j_1}) = [0, \beta)$ and $S(\alpha_{j_2}) = (2\pi - \beta, 2\pi]$ for some $\beta > 0$ (to be the only ones containing 0 and 2π). The β could be determined based on whether $h_i^{(2)}(0)$ is zero or not and the associated reminder term in the Taylors formula with reminder, by first extending h_i to a periodic function in $C^{2\nu+2}(\mathbb{R})$ since by assumption $h_i^{(k)}(0) = h_i^{(k)}(2\pi), k = 1, \cdots, 2\nu + 2$, at 0

(or equivalently at 2π). Then we could have chosen

$$\psi_{j_1}(x) = \begin{cases} g(x)e^{-(x-\beta)^{-2}}, & x \in [0, \beta) \\ 0, & \text{otherwise} \end{cases} \tag{33}$$

and

$$\psi_{j_2}(x) = \begin{cases} g(x)e^{-(x-2\pi+\beta)^{-2}}, & x \in (2\pi - \beta, 2\pi] \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

where $g(x)$ is the usual normalizing function to get the partition of unity.

We then consider the integral

$$\int_0^{2\pi} d\theta e^{-ith_i(\theta)+i(n-m)_i\theta}\{\psi_{j_1}(\theta) + \psi_{j_2}(\theta)\}d\theta$$

and it can be written as

$$\int d\theta e^{-ith_i(\theta)+i(n-m)_i\theta}\{\psi_{j_1}(\theta + 2\pi) + \psi_{j_2}(\theta)\}d\theta.$$

Then the function $\phi(x) = \psi_{j_1}(x)+\psi_{j_2}(x)$ is smooth and has support satisfying the assumptions of the proposition (4.4) so that we can get a bound of $C/|t|^{1/3}$ or $C/|t|^{1/2}$ for the above integral, based on the vanishing or otherwise of the second derivative of h_i at 0 (equivalently at 2π).

Lemma 3.4. *Let $\nu \geq 5$. Let S be a subset of $\mathbb{Z}^\nu$ satisfying $|S \cap \Lambda| \leq |\Lambda|^\alpha, \quad 0 < \alpha < 2(1/3 - 1/\nu)$, for any cube Λ as $|\Lambda| \to \infty$. Let H_0 be the operator associated with the function h satisfying the assumptions (3.1). Then*

1. *S is sparse relative to H_0.*
2. *If $\mathbb{D}$ denotes the set of vectors of finite support in $\ell^2(\mathbb{Z}^\nu)$, then for each t fixed $\|(1 + |m|)^\beta e^{-itH_0}\phi\| < \infty$, for $\nu + 1 > \beta > \nu$.*

Proof: To show that S is sparse relative to H_0, we consider

$$\|P_S e^{-itH_0}\phi\|$$

for ϕ such that $\langle\phi, \delta_k\rangle = 0$ for all but finitely many k and $\|\phi\| = 1$. Since H_0 has purely absolutely continuous spectrum, under assumption (3.1) on h,

this collection of ϕ forms a dense subset of the absolutely continuous spectral space of H_0. We show that this quantity is integrable in t ≥ 1, for all m and the integral is bounded by a constant independent of m and ϕ. We have

$$\begin{aligned}
\int dt \|P_S e^{-itH_0}\phi\| \leq \\
\|\phi\| \int dt \left(\sum_{m \in S} \left(\sum_{n:\phi(n)\neq 0} |\langle \delta_m, e^{-itH_0}\delta_n\rangle|^2 \right)\right)^{1/2} \\
\leq \|\phi\| \int dt \left(\left(\sum_{n:\phi(n)\neq 0} \sum_{m \in S} |\langle \delta_m, e^{-itH_0}\delta_n\rangle|^2 \right)\right)^{1/2} \\
\leq \|\phi\| \int dt \left(\sum_{n:\phi(n)\neq 0} \left(\sum_{m\in S:|n-m|>2\nu t\|h'\|} |\langle \delta_m, e^{-itH_0}\delta_n\rangle|^2 \right.\right. \\
\left.\left. + \sum_{m\in S:|n-m|\leq 2\nu t\|h'\|} |\langle \delta_m, e^{-itH_0}\delta_n\rangle|^2 \right)\right)^{1/2}
\end{aligned} \tag{35}$$

The last two summands are estimated using the two estimates of proposition (3.3), to get

$$\begin{aligned}
\int dt \|P_S e^{-itH_0}\phi\| \leq \\
\|\phi\| \int dt \left(\sum_{n:\phi(n)\neq 0} \left(\sum_{m\in S:|n-m|>2\nu t\|h'\|} C/|n-m|^{2\nu} \right.\right. \\
\left.\left. + \sum_{m\in S:|n-m|\leq 2\nu t\|h'\|} C/|t|^{2\nu/3} \right)\right)^{1/2} \\
\leq \|\phi\| \int dt \left(\sum_{n:\phi(n)\neq 0} \left(C/|t|^{\nu-} + C|t|^{\alpha\nu}/|t|^{2\nu/3} \right) \right)^{1/2} \\
< C\|\phi\| \#\{n : \phi(n) \neq 0\} < \infty,
\end{aligned} \tag{36}$$

in view of the assumptions on ν and α and the finiteness of the support of ϕ, with # denoting the cardinality of the set.

Part (2) is a direct consequence of the finiteness of the support of ϕ and the estimate in proposition (3.3)(1).

We take any subset S of $\mathbb{Z}^\nu$, satisfying the assumption in lemma (3.4). Consider for any k $\in \mathbb{Z}^+$, the function $h(\theta) = \sum_{i=1}^{\nu} 2\cos k\theta_i$, so that $H_0 = \sum_{i=1}^{\nu}(T_i^k + T_i^{-k})$, T_i denoting the shift by 1 in the i -th direction in $\mathbb{Z}^\nu$. Δ corresponds to $k = 1$. In this case S is H_0 sparse and in theorem (1.5) the mobility edges are $\{-2\nu, 2\nu\}$.

We presented in this paper a class of random operators, having both absolutely continuous spectrum and dense pure point spectrum. The a.c. spectrum seems to come from the fact that mostly the potential is zero, while the dense pure point spectrum seems to come from localization near the potential sites. The interesting aspect of the result is that there need not be any structure for S. One only requires that the set be asymptotically sparse. Our examples include cases where S is a subgroup of $\mathbb{Z}^\nu$, for large ν and then the results in this paper also have examples of ergodic potentials (with respect to S action) exhibiting the a.c spectrum and dense pure point spectrum.

The mobility edges are also identified for a class of potentials and a class of free operators provided the coupling constants go to infinity at infinity. In the paper of Kirsch-Krishna-Obermeit [16] we showed similar result for the case when the coupling constants decay to zero. Such examples in addition to sparseness also can be included here and the proof goes through for that case also.

4 Appendix:

In the appendix we collect a few results for the convenience of the reader.

In the paper [3] Aizenman-Molchanov introduced the decoupling principle, which was at the heart of their method of proving localization. The lower bounds that they obtain on the expected values of some random variables together with a uniform bound on the low moments on the Green functions of the problem, made the proof possible.

Their decoupling, stated in a version relevant for this paper is the following. The proof of this lemma is almost identical to the one when $S = \mathbb{Z}^\nu$, whose proof can be found in either Aizenman-Molchanov [3] Aizenman-Graf [2]. Nevertheless we present it for the convenience of the reader.

Proposition 4.1 (Aizenman-Molchanov). *Consider the operator H_λ^ω with $V_S^\omega(m)$ satisfying the assumptions(1.3). Then for any $\lambda > 0$, $0 < s < 1$, there is a positive constant $D(s)$ depending only upon μ and s, but bounded above and below as $s \to 1$, such that for each $m \in S$,*

$$\begin{aligned} &|\lambda|^s(1-s)^s D(s)\mathbb{E}\{|G(E+i\epsilon, n, m)|^s\} \\ &\leq \mathbb{E}\{|(\lambda V^\omega(m) - E - i\epsilon)G(+i\epsilon, n, m)|^s\}, \end{aligned} \tag{37}$$

for any $n \in \mathbb{Z}^\nu$.

Proof: First we note that our assumption on the measure μ (the distribution of the random variables $V_S^\omega(n)$ for any $n \in S$), is 1 regular in the sense of Aizenman-Molchanov [3]. Therefore the lemmas III.1 and III.2 of Aizenman-Molchanov [3], tell us that the measures $d\mu_s(x) = |x-\alpha|^s d\mu(x)$ are respectively are 1 and $d\mu_s(x) = |x-\beta|^{-s}d\mu(x)$ are respectively are 1 and (1-s) regular. Their proofs of the lemmas III.1 and III.2 applied to prove their lemma 3.1(i) show that we have the estimate,

$$\int |x-\eta|^s|x-\beta|^s d\mu(x) \geq (k_s)^s \int |x-\beta|^s d\mu(x)$$

with the constant $k_s = (1-s)/D(s)^s$, D(s) a constant depending only upon the measure μ, s but independent of $\eta, \beta \in \mathbb{C}$ and also bounded above and below as $s \to 1$. Using this estimate we see that for any real number γ,

$$\int |\gamma x-\eta|^s|\gamma x-\beta|^s d\mu(x) \geq |\gamma|^s(k_s)^s \int |\gamma x-\beta|^s d\mu(x)$$

Once this estimate is in place, the proof of the proposition is as in the proof of the decoupling lemma (2.3) of Aizenman-Molchanov [3]. Finally we remark that the resond we needed $m \in S$ is that otherwise $V_S^\omega(m) = 0$ and there is no random variable to integrate! This was essential in getting the uniform bounds on the energy E.

As an immediate corollary we see that

Proposition 4.2. *Consider the operator H_λ^ω with $V_S^\omega(m)$ satisfying the assumptions of theorem (1.5). Then there is a positive constant $D(s)$ depending only upon μ and s, but bounded above and below as $s \to 1$, such that for each $m \in S$,*

$$\begin{aligned} &|a_m|^s(1-s)^s D(s)\mathbb{E}\{|G(E+i\epsilon, n, m)|^s\} \\ &\leq \mathbb{E}\{|(\lambda V^\omega(m) - E - i\epsilon)G(+i\epsilon, n, m)|^s\}, \end{aligned} \tag{38}$$

for any $n \in \mathbb{Z}^\nu$.

Proof: This is an easy application of the proof of the earlier proposition where for each $m \in S$ instead of λ we now have a_m as the coupling constant.

Proposition 4.3. *Let B be a bounded self-adjoint operator, commuting with Δ and satisfying the assumption (1.1) for some $1 > s_0 > 0$. For each $s \in (s_0, 1)$, let $E \in [-\|H_0\|_s, \|H_0\|_s]$*

$$\sum_{m \in \mathbb{Z}^\nu} |(B - z)^{-1}(n, m)|^s \leq C(B, E, s) < \infty.$$

with $Re(z) = E$. The constant $C(B, E, s)$ is bounded as a function of E on any compact subset of $\mathbb{R} \setminus [-\|H_0\|_s, \|H_0\|_s]$.

Proof: Under the assumptions on z, it is in the resolvent set of B and the bounded operator $(B - z)^{-1}$ can be expanded using the Neumann series, which converges under the assumption on E = Re(z). Therefore we consider

$$(B - z)^{-1}(n, m) = \frac{1}{z} \sum_{k=0}^{\infty} \langle \delta_n, \frac{H_0^k}{z^k} \delta_m \rangle.$$

Taking the absolute values to the power s and using the inequality $\sum |x_i|^s \geq (|\sum x_i|)^s$, for the given s, we get that

$$|(B - z)^{-1}(n, m)|^s \leq \frac{1}{|z|} \sum_{k=0}^{\infty} |\langle \delta_n, \frac{H_0^k}{z^k} \delta_m \rangle|^s.$$

Therefore for any cube Λ centered at 0 in $\mathbb{Z}^\nu$, we have

$$\sum_{m \in \Lambda} |(B - E)^{-1}(n, m)|^s \leq \sum_{m \in \Lambda} \frac{1}{|z|} \sum_{k=0}^{\infty} |\langle \delta_n, \frac{H_0^k}{z^k} \delta_m \rangle|^s,$$

which implies

$$\sum_{m \in \Lambda} |(B - E)^{-1}(n, m)|^s \leq \frac{1}{|z|} \sum_{m \in \Lambda} \sum_{k=0}^{\infty} |\langle \delta_n, \frac{H_0^k}{z^k} \delta_m \rangle|^s.$$

On the other hand from the assumption on B we have that for any positive integer k,

$$\begin{aligned} \langle \delta_n, \frac{H_0^k}{z^k} \delta_m \rangle = \frac{1}{z^k} \sum_{l_1, \cdots, l_{k-1}} & \langle \delta_n, B\delta_{l_1} \rangle \langle \delta_{l_1}, B\delta_{l_2} \rangle, \cdots, \\ & \langle \delta_{l_{k-2}}, B\delta_{l_{k-1}} \rangle \langle \delta_{l_{k-1}}, B\delta_m \rangle. \end{aligned} \tag{39}$$

Therefore estimating after taking the sum over Λ we get, since $|z| \geq |E|$,

$$\sum_{m\in\Lambda} |\langle \delta_n, \frac{H_0^k}{z^k}\delta_m\rangle|^s = \frac{1}{|E|^{sk}} \sum_{m\in\Lambda} \sum_{l_1,\cdots,l_{k-1}} |\langle \delta_n, B\delta_{l_1}\rangle|^s |\langle \delta_{l_1}, B\delta_{l_2}\rangle|^s, \cdots, \\ |\langle \delta_{l_{k-2}}, B\delta_{l_{k-1}}\rangle|^s |\langle \delta_{l_{k-1}}, B\delta_m\rangle|^s. \tag{40}$$

This results in

$$\sum_{m\in\Lambda} |\langle \delta_n, \frac{H_0^k}{z^k}\delta_m\rangle|^s = \frac{1}{|E|^{sk}} \|B\|_s^{s(k-1)} \sup_{l_{k-1}\in\mathbb{Z}^\nu} \sum_{m\in\Lambda} |\langle \delta_{l_{k-1}}, B\delta_m\rangle|^s \\ \leq \frac{\|B\|_s^{sk}}{|E|^{sk}}. \tag{41}$$

This implies that

$$\sum_{m\in\Lambda} |(B-z)^{-1}(n,m)|^s \leq \frac{1}{|E|} \sum_{k=0}^{\infty} \frac{\|B\|_s^{sk}}{|E|^{sk}} < \infty,$$

under the assumptions on E, with the sum on the right denoted C(B, E, s). Since the bound on the right is independent of Λ, it is also valid for the supremum over all such Λ and by taking a collection of cubes increasing to $\mathbb{Z}^\nu$, we conclude the proposition.

We finally restate the proposition on stationary phase estimate from Stein [25], VIII.1.3., proposition 3. Below ϕ is a real valued function having (k+1) continuous derivatives in (a,b). and ψ is a smooth function whose support contains only one critical point of ϕ. We note that the assumptions on ϕ below allow us to approximate it by $(x-x_0)^k[\phi^{(k)}(x_0)+\epsilon(x)]$ with $\|\epsilon(x)\|_\infty \leq \phi^{(k)}(x_0)/2$ in the support of ψ, if it is small enough, using the Taylor's theorem with reminder. These are the conditions on ϕ and ψ required in the proof of the proposition below.

Proposition 4.4 (Stein). *Suppose $k \geq 2$, and*

$$\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0,$$

while $\phi^{(k)}(x_0) \neq 0$. If ψ is supported in a sufficiently small neighbourhood of x_0, then

$$I(\lambda) = \int \quad e^{i\lambda\phi(x)}\psi(x)dx \approx \lambda^{-1/k} \sum_{j=0}^{\infty} a_j \lambda^{-j/k},$$

in the sense that, for all integers N and r,

$$\frac{d^r}{dx^r}\left[I(\lambda)-\lambda^{-1/k}\sum_{j=0}^{N}a_j\lambda^{-j/k}\right]=O(\lambda^{-r-(N+1)/k})\quad as\quad \lambda\to\infty.$$

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